Dynamic Phase Transition in Prisoner's Dilemma on a Lattice with Stochastic Modifications.

M Ali Saif

Centre for Modeling and Simulation
University of Pune
Ganeshkhind, Pune, 411 007,
INDIA

E-mail: ali@cms.unipune.ernet.in

Prashant M Gade

Department of Physics
Rashtrasant Tukdoji Maharaj Nagpur University
Campus, Nagpur, 440 033
INDIA

E-mail: prashant.m.gade@gmail.com

Abstract. We present a detailed study of prisoner's dilemma game with stochastic modifications on a two-dimensional lattice, in presence of evolutionary dynamics. By very nature of the rules, the cooperators have incentive to cheat and the fear of being cheated. They may cheat even when not dictated by evolutionary dynamics. We consider two variants here. In either case, the agents do mimic the action (cooperation or defection) in the previous timestep of the most successful agent in the neighborhood. But over and above this, the fraction p of cooperators spontaneously change their strategy to pure defector at every time step in the first variant. In the second variant, there are no pure cooperators. All cooperators keep defecting with probability p at every time-step. In both cases, the system switches from coexistence state to an all-defector state for higher values of p. We show that the transition between these states unambiguously belongs to directed percolation universality class in 2+1 dimension. We also study the local persistence. The persistence exponents obtained are higher than ones obtained in previous studies underlining their dependence on details of dynamics.

Contents

1. Introduction

Cooperation is observed at many levels of biological organization. The evolution of cooperation in these systems has been a subject of extensive debate and studies[1]. Primarily, five different mechanisms have been proposed to explain how natural selection lead to cooperative behavior. They are kin selection, group selection, direct or indirect reciprocity and network reciprocity [2]. Kin selection explains the cooperation between genetically close organisms as a tendency to help reproductive success of the relatives even at a cost to themselves [3]. Cooperation may evolve not only on individuals level but also in groups. Thus, a group of cooperators are more likely to survive and grow than group of defectors. However, some authors believe that, the kin selection models are not different from the group selection models [4]. Cooperation is also observed between organisms who are not genetically close. Reciprocal altruism is a possible mechanism to explain the cooperation between such agents [5, 6]. Emergence of sustained cooperation when agents have an incentive to cheat as well as tension of being defected [7], has been a topic of extensive investigation. In this case, the benefit is extended to another organism in the hope that it will be reciprocated in future and this strategy is reversed if the act is not reciprocated. Sustaining such strategy is more likely in an iterated or spatial game theoretical model. We would like to mention that the cooperation is not always direct. Sometimes we help strangers, and there is no possibility for direct reciprocation. We would like to also mention that altruism is still an open problem.

Among various attempts at constructing a theory of cooperation, game theoretical models have played an important role [8]. In particular, Prisoner's Dilemma (PD) has emerged as a paradigm for the explanation of cooperative behavior among selfish individuals [9]. This kind of cooperative behavior observed in real life in systems ranging from biological to economic and social systems [10]. PD has now become a standard model to explain cooperation in these systems [6, 11, 12, 13]. In its original form, PD describes the pairwise interaction between two players. The player either cooperates(C) or defects(D) at any confrontation. If both players choose to cooperate (defect), they

get a pay-off of magnitude R (P) each; if one(D) chooses to defect, while the other(C) chooses to cooperate, the defector gets the biggest pay-off T, while the other gets S. For $T > R > P > S$ and $2R > T + S$, total reward for both players is higher if they cooperate. However, an individual has better payoff if he defects while the other player cooperates. Thus the best choice for any player is to defect irrespective of the opponent's choice if the game played for one round. However, on a two-dimensional lattice, it was found that a fraction of players keep cooperating with their neighbors with repeated interaction. *Thus, with repeated interactions and spatial structure, it was found that it is possible to have mixed state where clusters of cooperators coexist with defectors.* We must mention that recently other spatial structures have also received a fair share of attention [14, 15, 16, 17, 18, 19].

In the context of ecology, Nowak and May simulated PD game with choice of parameters $R = 1.0$, $T = b$ ($1.0 < b < 2.0$) and $S = P = 0.0$. (Some authors called this game 'weak dilemma' when $S = 0$ [8, 20]. However, Nowak and May [21] found that their qualitative results does not change when $S < 0$, at least for small absolute values of S, *i.e.* $|S| << 1$. Hence, we work with $S = 0$ in this paper. We have studied the case when $S = -0.01, -0.1$ and $S = -0.5$ to demonstrate that our main results do not change for $S < 0$.) They believe that, with this choice of parameters most of the interesting behavior is reproduced. They studied PD on a two dimensional array with synchronous updating and explored the asymptotic behavior for various values of the parameter b. Players interact with their local neighbors through simple deterministic rules and have no memory of past [21, 22].

This explanation was debated and robustness of the conclusions was studied under several perturbations of the model. Mukherji *et al.* as well as Huberman and Glance studied the system under introduction of asynchronicity [23, 24, 25]. Mukherji et al. investigated if cooperation can survive in the spatial PD in the presence of noise in general. They considered some more stochastic variants of this system. Other modifications by Mukherji *et al.* were random introduction of cooperators and defectors at any site and spontaneous conversion of cooperators into defectors with some probability [23]. Nowak *et al.* replied stating that, their results are robust with respect to these modifications. If one studies the entire parameter regime, cooperation is found to persist in the system even in presence of high values of noise [26]. We will make a detaied study of one of the cases studied by Mukherji *et al.* in which cooperators turn into defector spontaneously with probability p. We call this model as model stochastic prisoner's dilemma (permanent) (abbreviated as SPD(P)). Mukherji *et al.* simulated SPD(P) on a 100×100 lattice, for 500 generations with an initial condition of 90% cooperators. They found that, the density of cooperators quickly decreases with p and above certain value of p all agents become defor increase up to point where all players become defector[23]. This variant was criticized by Nowak *et al.* as 'this assumption is well chosen for attempting to eliminate cooperators' [26]. Hence, we will study one more variant, stochastic prisoner's dilemma (temporary) (SPD(T)). In this model, each cooperator turns into defector *temporarily* with probability p and

returns to being cooperator at the next time step. In both models, each agent imitate the best (unconditional imitation) strategy of neighboring agents in last time-step.

In this paper, we make an extensive study of SPD(P) and SPD(T) from the viewpoint of dynamic phase transition. In both cases, only one absorbing state is possible, namely, in which all players choose to defect at all times. Coexistence of D and C is considered as active phase or fluctuating phase. For large values of noise p, we observe a transition from coexistence state to an all-defector state. This kind of phase transition to absorbing state has attracted much attention recently [27, 28, 29, 30]. We can study this phase transition borrowing tools used extensively for studies in equilibrium systems, We find whether or not transition is continuous and find the several critical exponents. We also find the scaling functions which give a better idea of universality. At the basic level, the critical exponents allow us to classify the system in different universality classes. The concept of the universality is one of the most important concepts in study of phase transitions. It allows us to group different systems to small number of classes and lets us know the essential and not so essential details of the systems. It is generically believed that, all the continuous phase transitions from fluctuating phase to a single absorbing state are in the universality class of directed percolation (DP) [30]. (Most system with multiple absorbing state also fall in the DP class [31].) However, under some additional conditions, the systems with absorbing state may not fall under the DP class. The other well known universality class for such systems are parity conserving class [32, 33], the pair contact process with diffusion [34, 35], the conserved lattice gas [36, 37] and Manna class [38]. These systems has been studied extensively in past two decades [29, 30]. The SPD(P) and SPD(T) have an unique absorbing state, no obvious conservation laws, so we expect them to be in the DP universality class. We must mention that, phase transition to DP class has been observed in the PD game in a previous studies [39, 40, 41, 42]. However, there are a few differences between these works and the present one. The updating rule in these works was different from the updating rule suggested by Nowak and May. Most of earlier studies computed only the static exponent β. In this work, we have made exhaustive and systematic simulations and found all the three independent exponents of DP class. (In fact, we also find the fourth exponent and explicitly demonstrate time reversal symmetry.) Qualitatively, we have only one possible absorbing state in this system unlike previous studies where there are two absorbing states are possible. Thus there is a stronger ground for Janssen and Grassberger's, [30] conjecture to hold in this case. In the other hand, these transitions could be discontinuous. In fact, experimentally discontinuous transitions are observed often and unfortunately there are no clear thumb rules on when the transition is continuous and when it is not except in cases where mean field theory is applicable [43]. However, the transition in our case is clearly continuous.

In this paper, we make a complete study of transition to absorbing state of the above two models and we find that they are indeed in DP class. Furthermore, we study persistence in these systems. Recently, there have been several studies on persistence in dynamic phase transitions. They lead to nontrivial exponents which have

no obvious relation with other critical exponents in the system since persistence takes into account time correlations of arbitrary order. We study local persistence in these two models and determine the value of the local persistence exponent. Our studies further support the fact that the conjecture of superuniversality of the exponent [30] made in initial studies is not true. (Superuniversality means having an exponent which is independent of dimension.) In fact, the exponent is not even universal in the sense that different models in same universality class in same dimension yield widely different exponents. The exponents obtained by us are much larger than ones obtained in previous studies for some other systems belonging to DP class. With systematic numerical studies, we will demonstrate that the two variants studied by us are unambiguously in DP universality class. However, they show different persistence exponents. This is not entirely unexpected since then persistence exponent probes a full non-Markovian evolution of the system and is one of the least universal exponents.

The dependence of persistence exponent on detailed dynamics of the system has been observed previously in other systems such as spin systems [44]. Our results further demonstrate that, the persistence exponent is very much dependent on the dynamics and having the same exponent in two different systems could simply be a coincidence.

2. Definition of Models and Simulation Results

We consider evolutionary PD game on the two dimensional lattice of size L. Each lattice site can take only two values $s = 0$ (defector) or $s = 1$ (cooperator). We fix boundary condition and players have no memory of the past. The parameters are chosen to be $R = 1$, $S = P = 0$. We study the model under variation of $T = b$. Each player (i, j) interacts with his eight nearest neighbors (Moore neighbors) and himself. The total pay-off of any player $p_{(i,j)}(t)$ is the sum of the pay-offs from all nine interactions (with neighbors and self). In each Monte Carlo step, each player is allowed to update his strategy by adopting the strategy of the most successful neighbor. In SPD(P), after each Monte Carlo step, each cooperator may choose to change his state to defector state with probability p. In SPD(T), each cooperator defects with his neighbors with probability p, but unlike SPD(P), returns to cooperator status in the next time step even if temporary defection has delivered him a good payoff. If his payoff is lesser than any agent in the neighborhood (except himself), he mimics that neighbor's strategy *in previous timestep*. If the successful neighbor has cooperated (defected) in previous time step, he becomes cooperator (defector). In both cases, for $p = 0$ we recover the PD on the two dimensional lattice where agents are either pure cooperators or pure defectors. Furthermore, in both cases, the only possible absorbing state is an all-defector state for any value of $p \neq 0$ (We have also checked that, both models still show DP phase transition if we make $S < 0$ at least when $|S| << 0$.)

The order parameter in this case is the density of cooperators(active) sites $\rho(t) = \langle 1/N \sum_i s_i(t) \rangle$ where N is the total number of a lattice sites and $\langle ... \rangle$ denotes to the ensemble average. Clearly, this parameter has a nonzero value in the mixed state while

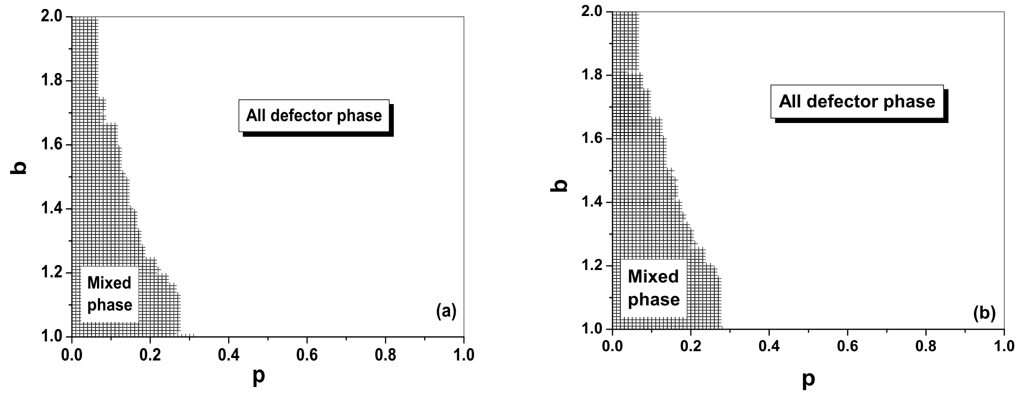

Figure 1. Schematic phase diagram of (a) SPD(P) and (b) SPD(T) when $S = 0$. The shaded area corresponds to an active phase and white area corresponds to an all-defector phase. We used lattice size $L = 60$ and averaged over 100 different initial samples after discarding 1000 time-steps

it is zero in an all-defector state. For both models SPD(P) and SPD(T), we plot a phase diagram for the asymptotic states in the phase space (p, b). We explore the range $0.0 < p < 1.0$ and $1.0 < b < 2.0$ for the two parameters. The corresponding phase diagram is shown in Fig.1. As we can see, the system has two different phase all defector phase (absorbing phase) and mixed phase (active phase). For fixed value of b, we vary p and study the nature of phase transition in the system.

First, we state results for SPD(P) model. We simulate the system on large enough lattice, *i.e.* on a lattice of size $L = 200$. We estimate the steady state of the order parameter ρ_{sat} by simulating system for very long time and confirming that the order parameter has reached its steady state. This procedure is carried out for various values of p. Initial condition consists of 30% defectors and 70% cooperators distributed randomly on the lattice sites. For every value of p, we average over 100 different initial configuration after discarding 10^5 timesteps near the critical point and 10^4 timesteps far from the critical point. In Fig. 2, we plot the average density of active sites ρ_{sat} as function of the control parameter p. It is clear that, the stationary density of the active sites varies continuously with p. The system crosses from absorbing phase to active phase at the critical point p_c. To determine the critical point p_c accurately, we use different lattice sizes up to $L = 512$. In all cases , we calculated the value of ρ_{sat} near the critical point after discarding 10^6 Monte Carlo steps. The best estimate of the value of a critical point in thermodynamic limit seems to converge to $p_c = 0.2708 \pm 0.0005$.

In order to confirm that, phase transition in these models is in DP class, we numerically determine the values of all the critical exponents. The absorbing phase transitions are characterized by four independent critical exponent β, $\acute{\beta}$, $\nu_\perp$ and $\nu_\parallel$. However, it is well known that, DP class displays a symmetry known as rapidity reversal symmetry. This implies that $\beta = \acute{\beta}$. (We must mention that this statement is easily proven for directed bond percolation. [45] It is not obvious for several other models.

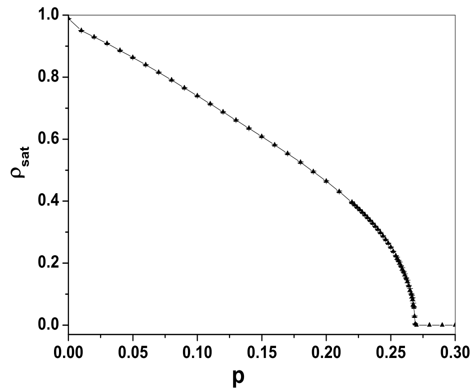

Figure 2. The steady state of the density of active sites at various value of parameter p for SPD(P) at $b = 1.05$.

Hence we explicitly checked this symmetry by computing all the exponents.) Thus, DP is characterized only by three critical exponent instead of four. All the other exponents can be expressed in terms of these exponents. The so called dynamic exponent z is given by $z = \frac{\nu_\parallel}{\nu_\perp}$. However, the exponents δ, α and θ are given by $\delta = \frac{\acute{\beta}}{\nu_\parallel}$, $\alpha = \frac{\beta}{\nu_{|}}$ and $\theta = d/z - 2\delta$ (for more details see [29, 30] and the references therein). We would like to emphasize that, in this work we have verified the equality of β and β' by computing survival probability $P(t)$ as well as density of active sites $\rho(t)$ independently. If $\beta = \beta'$, both would decay with the same exponent. By finding effective exponents for $P(t)$ and $\rho(t)$, and showing that they are equal, we have verified this symmetry for these models (See Fig. 4 and 7.)

It is known that, for continuous phase transition, the stationary value of order parameter ρ_{sat} vanishes as the control parameter p approaches a critical value p_c asymptotically according to a power-law as follows:

$$\rho_{sat} \sim (p_c - p)^\beta \tag{1}$$

The value of exponent β can be found by plotting the value of ρ_{sat} as a function of $(p - p_c)$ on a logarithmic scale Fig. 3. The power-law behavior is clear and the best-fit value of the critical exponent is found to be $\beta = 0.57 \pm 0.01$ which matches very well with value of $\beta = 0.58$ in the DP class [30]. The compatibility of this exponent with the DP in the $2 + 1$ dimension, leads us to conjecture that, SPD(P) belongs to the directed percolation universality class in $2 + 1$ dimension.

To be sure about the universality class, we extract further critical exponents. Finding the nature of phase transitions is an 'asymptotic' game, in the sense that we need to make conjectures about asymptotic behavior of thermodynamic system by systematically simulating systems of finite size for finite time. Fortunately some of the information about nature of transition can be inferred from short time dynamics. Thus it is a simplest numerical method which allows us to estimate some of the critical

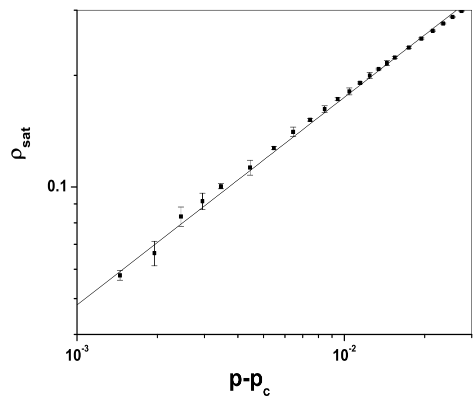

Figure 3. Stationary density of active sites ρ_{sat} is plotted as a function of the distance to the phase transition in log-log scale for SPD(P). The linear fit accurately fits to the numerically obtained data with the exponent $\beta = 0.57 \pm 0.01$.

exponents. We start Monte Carlo simulation with a fully occupied lattice [30, 46, 47]. At the critical point p_c, the order parameter $\rho(t)$ decays asymptotically according to a power-law

$$\rho(t) \sim t^{-\delta} \tag{2}$$

In Fig. 4a and 4b, we plot $\rho(t)$ as a function of time t in a logarithmic scale for both the models. At the critical point, the order parameter $\rho(t)$ shows a power-law decay. The best fit of the critical exponent is $\delta = 0.456 \pm 0.001$ for SPD(P) and $\delta = 0.434 \pm 0.002$ for SPD(T), which again is in good agreement with the value $\delta = 0.451$ in $2+1$ dimensional class [30]. In Fig. 4, we display $\rho(t)$ as a function of t for $p < p_c$ and $p > p_c$ also. As expected, the density of active sites go to zero (absorbing state) for $p > p_c$ while this density saturates to some asymptotic value signaling the presence of coexistence phase for $p < p_c$.

In addition, the nonequilibrium phase transitions are characterized by two independent correlation length, spatial length scale $\xi_\perp$ and a temporal length scale $\xi_\parallel$. Close to the transition point, these length scales are expected to diverge as:

$$\xi_\perp \sim |p - p_c|^{-\nu_\perp}, \xi_\parallel \sim |p - p_c|^{-\nu_\parallel} \tag{3}$$

The two correlation lengths are related by $\xi_\parallel \propto \xi_\perp^z$ where z is the dynamic exponent. In order to obtain the dynamic exponent and the two correlation exponents, we carry out the off-critical simulations and finite size scaling. For DP, particles density $\rho(t)$ starting from fully occupied lattices is expected to scale with time and lattice size as follows [30]:

$$\rho(t) \sim t^{-\beta/\nu_\parallel} f(\Delta t^{1/\nu_\parallel}, t^{d/z}/N) \tag{4}$$

where $\Delta = |p - p_c|$ and $N = L^d$ is the total number of sites. The exponent δ is given by $\delta = \beta/\nu_\parallel$. By plotting the value of $\rho(t)t^\delta$ versus $t\Delta^{\nu_\parallel}$ for different values of Δ we

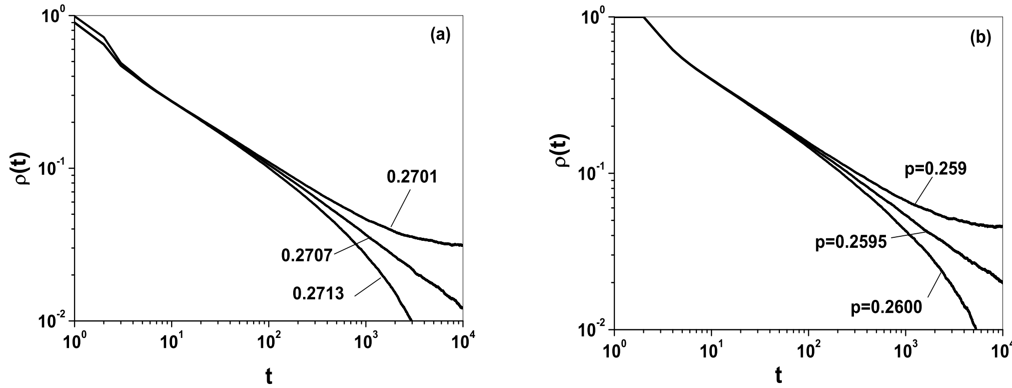

Figure 4. The dynamical behavior of the density of active sites as function of time (a) for SPD(P) and (b) for SPD(T), for lattice size $L = 512$. The data averaged over 10^3 samples

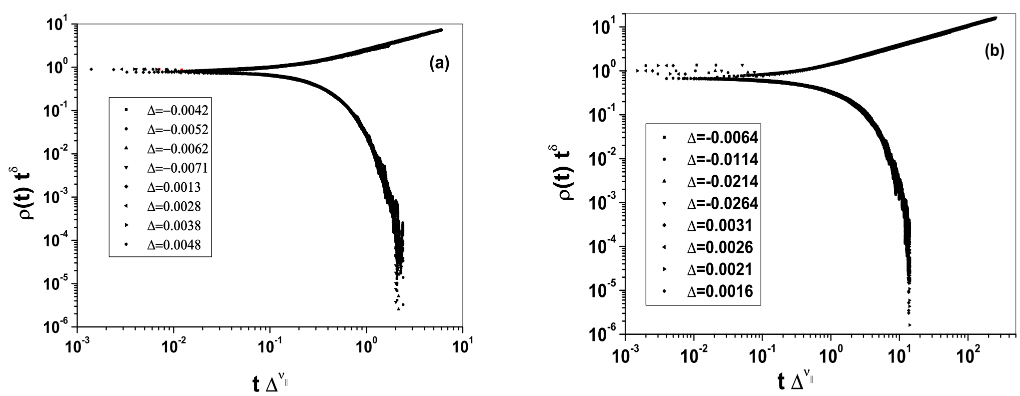

Figure 5. The off-critical scaling function of the density of active sites (a) for SPD(P) and (b) for SPD(T). The curves collapse according to the scaling form Eq. (4).

can tune the exponent $\nu_{||}$ such that all curves collapse on single curve. In Fig. 5, we found the best collapse is achieved for $\nu_{||} = 1.295 \pm 0.003$ for SPD(P), for the SPD(T) $\nu_{||} = 1.295 \pm 0.004$.

In these simulations, our lattices are large enough so that finite size effects are not very prominent. However, as in case of equilibrium scaling, we can carry out finite size scaling in DP to find further a critical exponents. The system size enters here as an additional scaling field. At the critical point, the finite size simulations can yield us the value of dynamic exponent z (See Eq. (4)). We have plotted the $\rho(t)t^{\delta}$ versus $t/N^{\frac{z}{d}}$ for different system size Fig. 6 at $p = p_c$. By tuning the value of exponent z, the best collapse is obtained for $z = 1.76 \pm 0.03$ for SPD(P) and $z = 1.76 \pm 0.02$ for SPD(T) which matches with $z = 1.76$ for DP in 2+1 dimensions [30]. Thus three independent exponents $\beta = \delta\nu_{||}$, δ and z match well with DP in 2+1 dimensions for

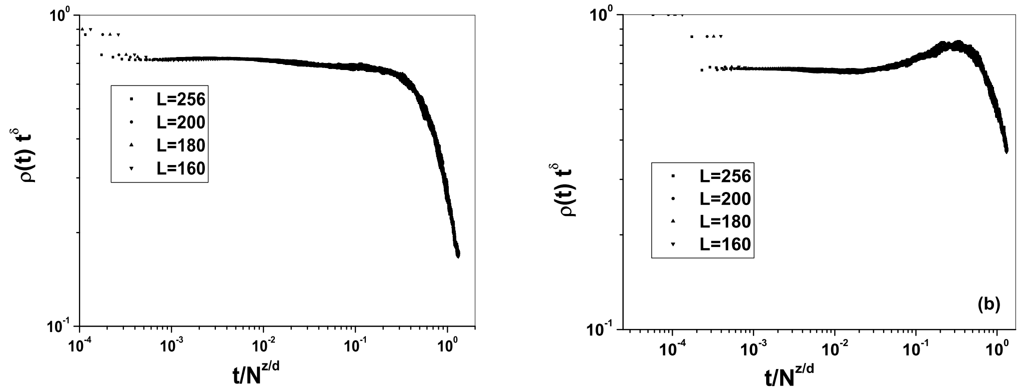

Figure 6. We demonstrate finite-size scaling density of active sites at the critical point for various values of lattice sizes (a) for SPD(P) and (b) for SPD(T). An excellent collapse is obtained according to the scaling form Eq. (4).

SPD(P) and SPD(T). Other exponents can be found from these exponents and agree well with values in literature. For example, the exponent z related to the temporal and spatial correlation exponents with that relation $z = \nu_{\|}/\nu_{\perp}$. Thus value of $\nu_{\perp} = 0.7358$ for both models which matches with value quoted in literature.

To confirm the results, we make more accurate estimates. These can be obtained by dynamic simulations starting from a configuration which is close to the absorbing state. (In our system, we cannot start from a single active site which will disappear immediately.) We start our simulation from five active sites are located in the center of lattice. We distributed these sites as follows; we put one site in the center of lattice and the other four active sites as the first neighbor of that centered site. In this case, there is a possibility for the sites in centre to survive and grow. We use the time-dependent simulations [48] to estimate values of θ and δ (or z) and confirm previous results. We follow the time evolution of this system which is initially very close to the absorbing state [49, 50]. We numerically measure the survival probability $P(t)$ (the probability that the system does not reach the absorbing state till time t), the average number of active sites $n(t)$, and the average mean square distance of spreading of active sites from the origin $R^2(t)$. At the critical point these quantities are expected to display asymptotic power-laws:

$$P(t) \sim t^{-\delta} \tag{5}$$

$$n(t) \sim t^{\theta} \tag{6}$$

and

$$R^2(t) \sim t^{2/z} \tag{7}$$

To determine the critical exponents more accurately, we adopted the local slope

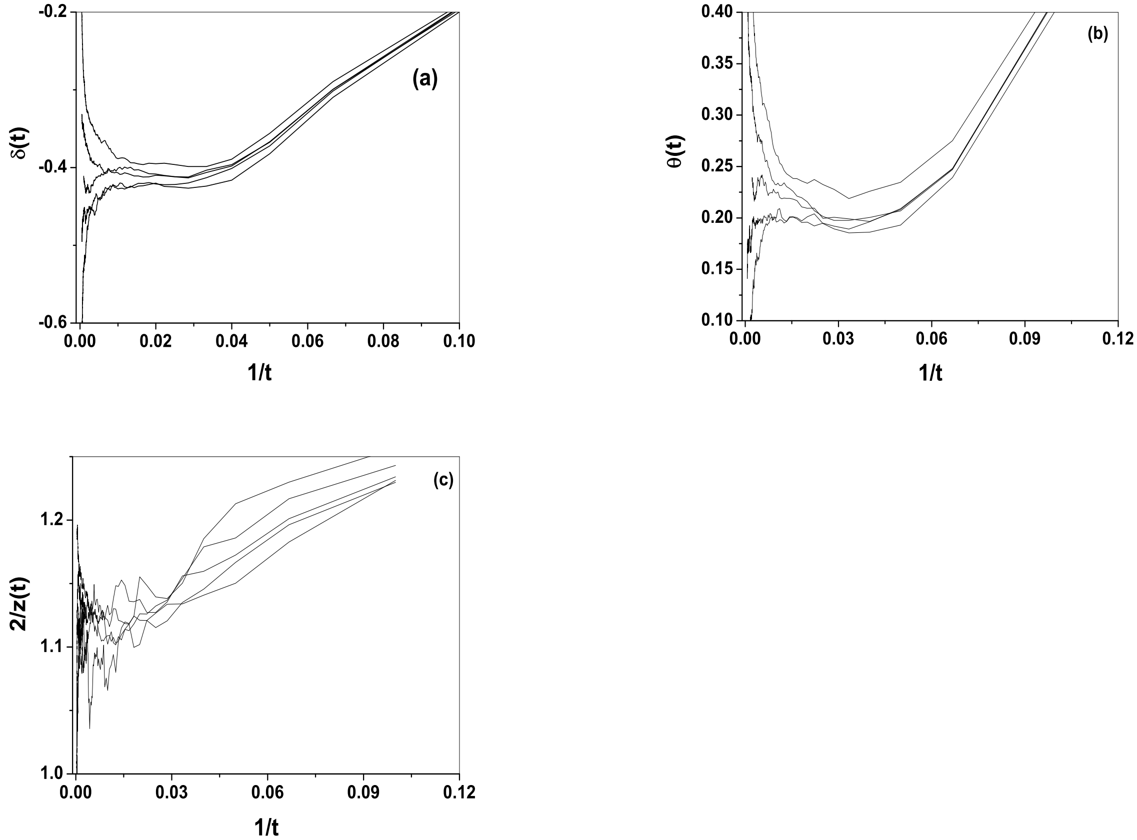

Figure 7. Time dependent behavior of the effective exponents (a) $\delta(t)$, (b) $\theta(t)$ and (c) $2/z(t)$ as function of $1/t$ for the value of $p = 0.2700, 0.2705, 0.2708, 0.2711$ and 0.2714 (from top to bottom curves) for SPD(P).

method by introducing the *effective exponent* [30, 51], as follows:

$$-\delta(t) = \frac{\log_{10}(P(t)/P(t/m))}{\log_{10} m} \tag{8}$$

where m is a fit parameter. we can get similar definitions for the effective exponents for the quantities $\theta(t)$ an $2/z(t)$. As $t \to \infty$ we should get the right value of the critical exponent.

We use $L = 512$ in our simulations and average over 1.2×10^4 initial conditions. We fix $m = 5$. In Figs. 7(a), (b) and (c), we show the values of effective exponents δ, θ and $2/z$ as a function of $1/t$. For $p \neq p_c$ the values tend to zero or escape to infinity while they tend to a constant value only for $p = p_c$. The estimated values $\delta = 0.434 \pm 0.005$, $\theta = 0.232 \pm 0.004$ and $2/z = 1.114 \pm 0.003$ are in excellent agreement with the exponents for DP in 2+1 dimensions within the error bars.

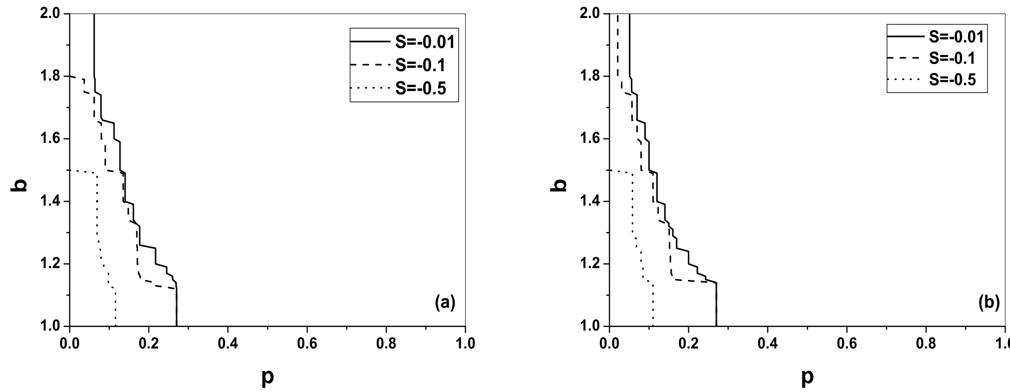

Figure 8. Schematic phase diagram of (a) SPD(P) and (b) SPD(T) for $S = -0.01, -0.1$ and -0.5. The area left to the curves correspond to an active phase whereas the area right to the curves corresponds to an all-defector phase. We used lattice size $L = 60$ and averaged over 50 different initial samples after discarding 1000 time-steps

2.1. A case of $S < 0$

Some authors believe that the case $S = 0$ corresponds to weak dilemma and we have prisoner's dilemma only for $S < 0$. We demonstrate that, our main conclusions remain unchanged for $S < 0$. In Fig. 8, we present the phase diagram for few negative values of S, namely, $S = -0.01, -0.1$ and even -0.5 for both models SPD(P) and SPD(T). For $S = -0.01$, the phase diagram does not change in a significant manner from $S = 0$. However, as one would expect, the area of parameter space which allows mixed state shrinks with decreasing values of S.

For $b = 1.05$, we find the critical parameter value of p for different values of S. For SPD(P), $p_c = 0.268$ for $S = -0.01$ and $p_c = 0.1318$ for $S = -0.5$. For SPD(T), $p_c = 0.257$ for $S = -0.01$ and $p_c = 0.125$ for $S = -0.5$. We have plotted the density of active sites $\rho(t)$ as a function of time at the critical parameter values in Fig. 9. We clearly see a power law decay of active sites for $t > 1$. The best fit of the critical exponent for SPD(P) in these cases are $\delta = 0.432 \pm 0.001$ for $S = -0.5$ and $\delta = 0.431 \pm 0.001$ for $S = -0.01$. For SPD(T) $\delta = 0.431 \pm 0.002$ when $S = -0.5$ and $\delta = 0.440 \pm 0.001$ when $S = -0.01$. These values match well with the known value of δ for DP in $2 + 1$ dimensions. Thus it is clear that the transition remains in DP universality class even for negative values of S.

2.2. 1-D and 3-D case

PD in 1-D system, when the player i interact with his first two neighbors without self-interaction leads to absorbing state for each value of $T > 1$. Hence, there is no phase transition in this case. However, preliminary investigation of SPD(P) and SPD(T) in 3-D suggest that, both of these model indeed have phase transition which falls in the

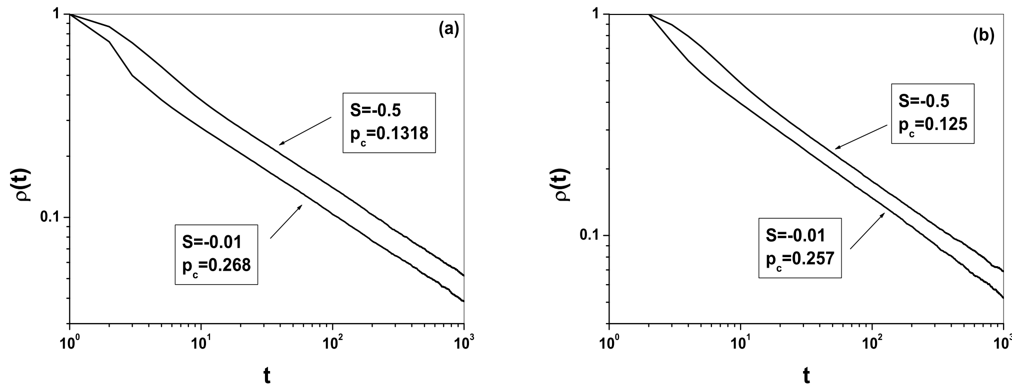

Figure 9. The dynamical behavior of the density of active sites as function of time at the critical point when $S = -0.01$ and $S = -0.5$ (a) for SPD(P) and (b) for SPD(T), for lattice size $L = 100$. The data averaged over 3×10^3 samples.

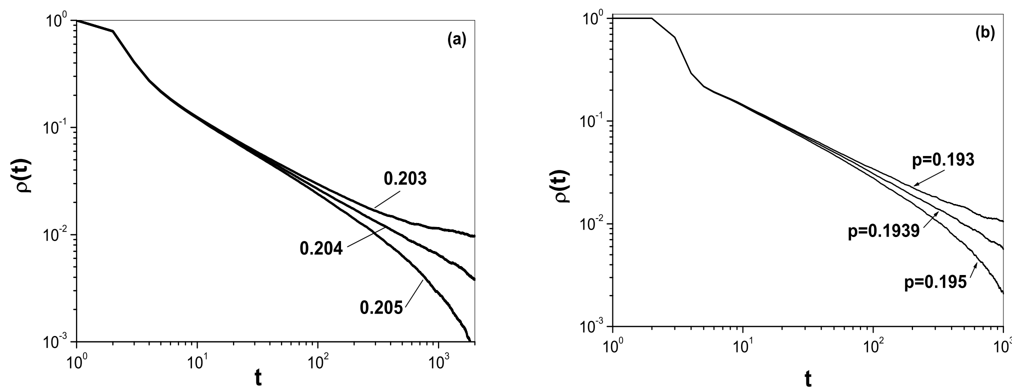

Figure 10. The dynamical behavior of the density of active sites as function of time in three dimensional case (a) for SPD(P) and (b) for SPD(T), for lattice size $L = 60$. The data is averaged over 200 samples

universality class of DP. In Fig.10, we plot the density of active sites $\rho(t)$ as function of time for both models. We use lattice size $L = 60$, temptation value $T = 1.1$ and each player i interact with his 6 nearest neighbors without self-interaction. At the critical point, the order parameter $\rho(t)$ displays a power-law decay. The best fit of the critical exponent is $\delta = 0.718 \pm 0.008$ for SPD(P) and $\delta = 0.723 \pm 0.005$ for SPD(T) which is in reasonable agreement with the value of $\delta = 0.73$ in $3 + 1$ dimension. We expect the transition to be in DP universality class for higher dimensions as well.

3. Local Persistence

Recently, persistence has been a fairly popular topic and has been investigated in great detail in statistical physics. While most of the studies are theoretical

[52, 53, 54, 55, 56, 57, 58, 59, 60, 61], there have been a few experimental studies as well [62, 63, 64]. It has been shown that, the persistence exponent is a rather nontrivial quantity to compute even in the simplest of the cases. One needs to know time correlation at all times and knowing it in the asymptotic limit is not good enough. Various definitions of persistence such as local persistence, global persistence, block persistence etc. have been proposed [65]. Though main object of studies has been discrete systems, the definition has been suitably modified and studied also for continuous systems such as coupled maps [66, 67, 68]. The most widely studied quantity in this context is the local persistence probability $p_l(t)$. It is defined as the probability that a local variable at a given point of space has not changed its state until time t during stochastic evolution. It is observed that, in several systems, at the critical point, the local persistence probability decays algebraically as follows:

$$p_l(t) \sim t^{-\theta_l} \tag{9}$$

where θ_l is the local persistence exponent. This exponent is found to be independent one in the sense that it cannot be obtained from other critical exponents. There are no scaling relations to link it with other exponents. In some cases, different models displaying continuous transition belonging to same universality class show the same exponent. For example, the Domany-Kinzel (DK) automata in one dimension and coupled circle maps in one dimension show transition to absorbing state which is in DP universality class and they show the same exponent [66]. However, in general, since the persistence exponent probes the full evolution of the underlying systems, it may not be the same in different systems. While we have shown above that the systems under study unambiguously display a dynamic phase transition in DP universality class, persistence exponents do not really match the systems studied previously.

The definition of persistence has to be appropriately modified for absorbing state transitions. Hinrichsen and Koduvely argued that, the previous definition of local persistence not appropriate for the DP class systems. They define the local persistence $p_l(t)$ as the probability that inactive site does not become active up to time t. (The simulations are started from random initial conditions.) The reason for this slightly changed definition is the asymmetry between active and inactive sites in absorbing states models. (The active sites can spontaneously turn into inactive sites. Thus number of active sites which do not become inactive even once till time t decays exponentially. On the other hand, a given inactive site may remain inactive for a very long time [69] and will stay so unless it comes in contact with an active site.) We follow the same definition for persistence in this work. It is reasonable in our system since cooperators (active sites) keep defecting with probability p leading to exponential decay, while defectors (inactive sites) may stay so for really long time.

Local persistence exponent θ_l in the different DP systems on $1+1$ dimension is found to be approximately $\theta_l = 1.5$ [69, 70, 71, 73]. However, in the case of the $2+1$ dimension there is no exact estimate of the value θ_l. In the table I, we tabulate the values of θ_l for the different systems showing DP transition in $2+1$ dimension.

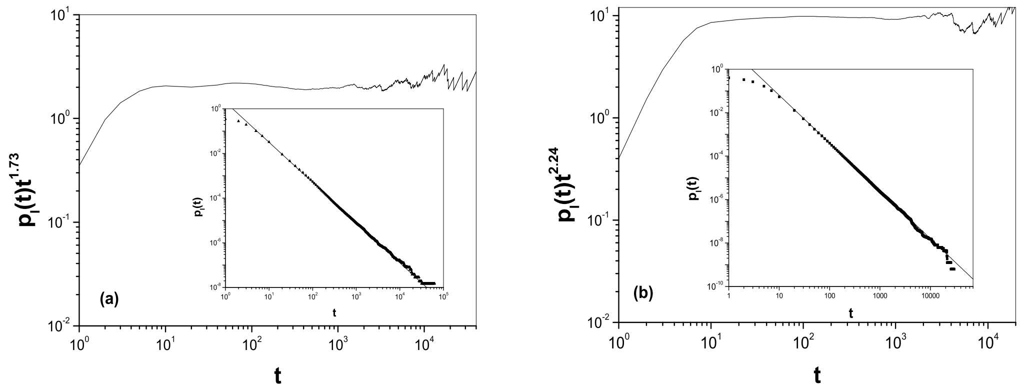

Figure 11. The local persistence probability is plotted as a function of t for, (a) SPD(P) model at p_c. The best fit for power-law decay at the critical point has a slope $\theta_l = 1.73$. (b) SPD(T) model at p_c. The best fit for power-law at the critical point has a slope $\theta_l = 2.24$. In both cases, the lattice size $L = 2000$ and we average over 100 different initial conditions.

Table I: The value of θ_l in DP in 2+1 dimensions

Model	ZGP	CP	CP	Bond-DP	SPD(P)	SPD(T)
Ref.	[70]	[71]	[72]	[71]	This work	This work
d=2	1.50(1)	>1.62	1.611(1)	>1.58	1.73 $\pm$0.02	2.24 $\pm$0.03

We carried out simulations at critical point for $L = 2000$ and averaged over 100 independent runs. Initial condition consists of 35% defectors distributed randomly on the lattice sites. In Fig. 11, we clearly observe that the number of persistent sites $p_l(t)$ decays as a power-law at the critical point for both SPD(P) and SPD(T) models for three decades . The best power law fit is obtained for $\theta_l = 1.73$ for SPD(P) and $\theta_l = 2.24$ for SPD(T). We also plot $p_l(t)t_l^{\theta}$ as a function of t and the curve is flat for almost three decades or more. We have estimated θ_l in two more ways. We carried out extensive simulations for $L = 1024$. We carry out effective exponent analysis and also scaling for off-critical simulations. The effective exponent analysis is presented in Fig. 12. We make a fit as suggested in original work by Grasberger (eq. 9 of ref. [51]) at the critical point and get an estimate of the value in the limit $1/t \to 0$. The fluctuations at large times for persistence (due to smaller data and finite size effects) are reflected in in the limit $1/t \to 0$ for effective exponent. When we make a fit suggested by Grassberger for effective exponent, we obtain $\theta_l = 1.72 \pm 0.01$ for SPD(P) and $\theta_l = 2.25 \pm 0.02$ for SPD(T).

The above exponents are confirmed by studying the scaling behavior of the local persistence probability. In analogy to other DP quantities, the local persistence is expected to have a scaling law of the form [71]:

$$p_l(t) \sim t^{-\theta_l} F(t\Delta^{\nu_\parallel}) \tag{10}$$

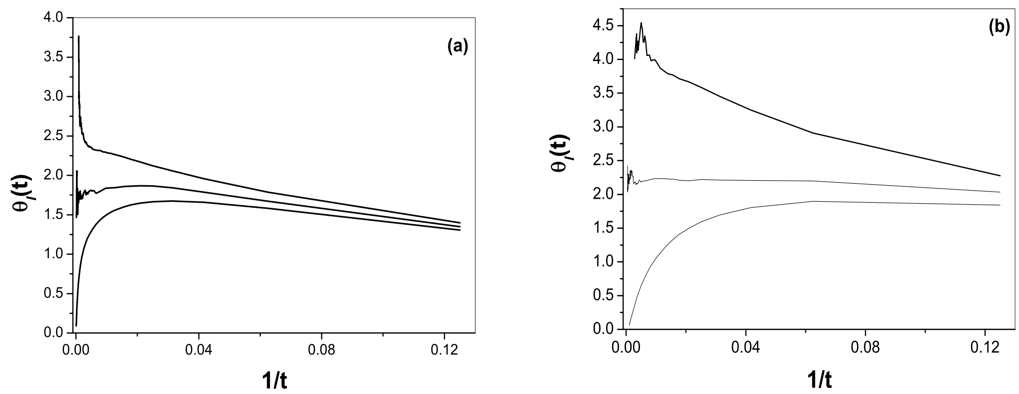

Figure 12. Time dependent behavior of the persistence exponents $\theta_l(t)$ as function of $1/t$ (a) for SPD(P) at the values of $p = 0.2620$, 0.2704 and 0.2740 (from top to bottom curves) (b) for SPD(T) at the values of $p = 0.2400$, 0.2584 and 0.2600 (from top to bottom curves).

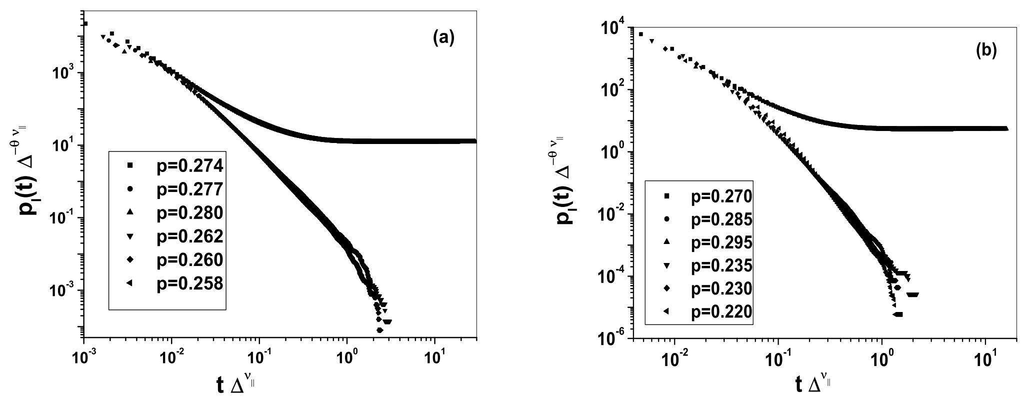

Figure 13. The off-critical scaling function of the local persistence near the critical point for $L = 1024$. We average over 100 different initial conditions. (a) SPD(P) model: the best collapse is obtained when $\theta_l = 1.73$ for SPD(P). (b) SPD(T) model: the best collapse is obtained when $\theta_l = 2.24$.

where $\Delta = |p - p_c|$ measures the distance from the critical point, F is the off-critical scaling function and $\nu_\parallel = 1.295$ is the temporal dynamical exponent of DP.

In the Fig. 13, we have plotted the value of $p_l(t)\Delta^{-\theta_l \nu_\parallel}$ against $t\Delta^{\nu_\parallel}$ for various values of the parameter p. The curves shows us a good collapse when the value of $\theta_l = 1.73 \pm 0.02$. Similarly, for SPD(T), the best collapse is obtained for $\theta_l = 2.24 \pm 0.03$. Hence, we conclude that the best estimates for persistence exponent are 1.73 ± 0.02 for SPD(P) and 2.24 ± 0.03 for SPD(T). It could be noted that the values of these exponents are much higher than those obtained in 2-d directed percolation in previous studies.

4. Conclusions

Good models for realistic situations in ecological and social systems require robustness with respect to some degree of noise. The game theoretical models with stochastic modifications are relatively less studied and in this paper we have tried to investigate two such models in detail. We have studied their phase diagrams and also studied the nature of dynamic transition between the two phases observed in these systems. In particular, we have studied two stochastic variants of prisoner's dilemma (SPD), (SPD(T) and SPD(P)), on a two dimensional lattice. Our investigations indeed confirm that the results from original model, prisoner's dilemma on a lattice, are reasonably robust with respect to noise. In the models we studied, the cooperators turn defectors temporarily or permanently. While SPD(P) was studied previously, SPD(T) is introduced by us in this work. The difference between these models is the following: In SPD(P) model, the cooperators spontaneously become defectors with probability p and stay so unless a cooperator in vicinity has higher payoff. On the other hand, in SPD(T) the defect temporarily for one timestep. In both the models, depending on value of parameter p, the system is found to be in the mixed phase or an all-defector phase. The memory is of one timestep only and the neighboring site does not distinguish between pure defector and a cooperator who has temporarily turned a defector. The phase diagrams are studied in detail and for a higher tendency of cooperators to defect, the mixed systems breaks down and we have an all-defector state. This is clearly an absorbing state transition.

We have carried out heavy and systematic computation on this system and a clear evidence has been presented that both SPD(P) and SPD(T) display a transition in the DP universality class. All the DP exponents have been found and a clean scaling behavior is presented in both cases. Of late, persistence in spatially extended dynamical systems has been a topic of intensive studies in nonequilibrium statistical physics. Systems displaying phase transition in universality class of directed percolation have also been studied in this regard and the persistence exponent in two dimensions is found to be in the range $1.5 - 1.6$. The persistence exponents in our systems are found to be significantly higher and in one of the cases the exponent is well beyond two. This should put any possible speculation about superuniversality of this exponent to rest. In this case, a clean scaling behavior is presented which demonstrates the validity of conventional scaling. To the best of our knowledge such clean scaling of persistence has not been shown in $2 + 1$ dimensions. As found in spin systems, the persistence exponent is the least universal of critical exponents [66]. For example, let us consider Ising model at finite temperature. Though the persistence probability shows the same behavior whether heat bath algorithm is employed or Glauber algorithm is employed for temperature $T < T_c$, it is very different for temperatures $T \geq T_c$, where T_c is the critical temperature [44]. However, it is an interesting fact that this quantity displays a power law at the critical point. In some cases, the exponent has been found theoretically. However, it is still a puzzle what this exponent means physically. For example, it is not clear which new physical insight is brought by having the information that the

persistence exponent 0.1207 for 1-D diffusion, a problem which is fully understood and is exactly solvable [65].

There are several other variants which could be an object of studies in future. If we consider cooperation with probability p and defection as two possible strategies and if we impose a condition that the strategies are mimicked and not behavior in previous time step, this variant can have two possible absorbing state and an interesting phase diagram. We will be investigating this variant in future studies. One could also consider dynamic phase transitions in presence of random introduction of defectors and cooperation as well as effect of asynchronicity. Nature of dynamic phase transitions in game-theoretic systems is a rich and unexplored field and it could yield interesting insights. In this work, we have brought out two models which are unambiguously in the universality class of directed percolation. Though their critical exponents match with standard DP exponents in $2+1$ dimensions, they have widely different persistence exponents. Thus having same exponent in two different systems as in coupled map lattice in one dimension and DK automata as pointed out by Menon *et al.* [73] could be a coincidence or presence of certain dynamical properties which needs further investigation.

5. Acknowledgment

MAS thanks Govt. of Yemen for scholarship and PMG thanks DST for financial support.

References

[1] Nowak M A, 2006 *Evolutionary Dynamics: Exploring The Equation of Life* (The Belknap Press of Harvard University Press, Cambridge)
[2] Nowak M A, 2006 Science **314** 1560
[3] Hamilton W D, 1963 American Naturalist **97** 354
[4] Lehmann L, Keller L, West S and Roze D, 2007 Proc. Nat. Acad. Sci. **104** 6736 and references therein
[5] Trivers R L, 1971 Q. Rev. Biol. **46** 35
[6] Axelrod R and Hamilton W D, Science 1981 **211** 1390
[7] Macy M W and Flache A, 2002 Proc. Nat. Acad. Sci. **99** 7229
[8] Szabó G, Fáth G, 2007 Phys. Rep. **446** 97
[9] Axelrod R, 1997 *The Complexity of Cooperation: Agent-Based model of competition and collaboration* (Princeton, New Jersey,Princeton University Press)
[10] Turner P E and Chao L, 2003 Am. Nat. **161** 497
[11] Turner P E and Chao L, 1999 Nature **398** 441
[12] Nowak M A and Sigmund K, 2004 Science **303** 793
[13] Doebeli M and Hauert C, 2005 Ecology Lett. **8** 748
[14] Abramson G and Kuperman M, 2001 Phys. Rev. E **63** 030901
[15] Szabó G, Vukov J and Szolnoki A, 2005 Phys. Rev. E **72** 047107
[16] Vukov J and Szabó G, 2005 Phys. Rev. E **71** 036133
[17] Kim B J, Trusina A, Holme P, Minnhagen P, Chung J S, and Choi M Y, 2002 Phys. Rev E **66** 021907
[18] Vukov J, Szabó G and Szolnoki A, 2006 Phys. Rev. E **73** 067103
[19] Wang S, Szalay M S, Zhang C and Csermely P, 2008 PloS ONE **3** e1917
[20] Alonso J, Fernández A and Fort H, 2006 J. Stat. Mech.: The. Exp. **6** 06013

[21] Nowak M A and May R M, 1992 Nature **359** 826
[22] Nowak M A, Bonhoeffer S and May R M, 1994 Int. J. Bif. Chaos **4** 33
[23] Mukherji A, Rajan V and Slagle J R, 1996 Nature **379** 125
[24] Huberman B A and Glance N S, 1993 Proc. Nat. Acad. Sci. **90** 7716
[25] Saif M A and Gade P M, 2009 J. Stat. Mech.: The. Exp. P07023
[26] Nowak M A, Bonhoeffer S and May R M, 1996 Nature **379** 126
[27] Marro J and Dickman R, 1999 *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England)
[28] Ódor G, 2004 Rev. Mod. Phys. **76** 663
[29] Lübeck S, 2004 Int. J. Mod. Phys. B **18** 3977
[30] Hinrichsen H, 2000 Adv. Phys. **49** 815 and references therein.
[31] Marques M C and Mendes J F F, 1999 Eur. Phys. J. B **12** 123
[32] Takayasu H and Tretyakov A Y, 1992 Phys. Rev. Lett. **68** 3060
[33] Jensen I, 1994 Phys. Rev. E **50** 3623
[34] Ódor G, 2000 Phys. Rev. E **62** R3027
[35] Hinrichsen H, 2001 Phys. Rev. E **63** 036102
[36] Park K, Kang S and Kim I M, 2005 Phys. Rev. E **71** 066129
[37] Lee S -G and Lee S B, 2008 Phys. Rev. E **77** 021113
[38] Manna S S, 1991 J. Phys. A **24** L363
[39] Szabó G and Toke C, 1998 Phys. Rev. E **58** 69
[40] Chiappin J R N and Oliveira M J de, 1999 Phys. Rev. E **59** 6419
[41] Hauert C and Szabó G, 2005 Am. J. Phys **73** 405
[42] Guan J Y, Huang Z X, Xu X J and Wang Y H, 2006 Eurphys. Lett. **76** 1214
[43] Monetti R A, 2001 Phys. Rev. E **65** 016103; Bidaux R, Boccara N and Chaté H, 1989 Phys. Rev. A **39** 3094
[44] Cueille S and Sire C, 1999 Euro. Phys. J. B **7** 111
[45] H. Hinrichsen , 2006 Physica A **369** 1
[46] Silva R da, Dickman R and Felício J R D de, 2004 Phys. Rev. E **70** 067701
[47] Lübeck S, cond-mat/0501259v1 (2005).
[48] Grassberger P and Torre A de la, 1979 Ann. Phys. (N. Y.) **122** 373
[49] Jensen I, 1991 Phys. Rev. A **43** 3187
[50] Dickman R, 1996 Phys. Rev. E **53** 2223
[51] Grassberger P, 1989 J. Phys. A: Math. Gen. **22** 3673
[52] Majumdar S N, Sire C, Bray A J, and Cornell S J, 1996 Phys. Rev. Lett. **77** 2867
[53] Derrida B, Bray A J, and Godrèche C, 1994 J. Phys. A **27** L357
[54] Krapivsky P L, Ben-Naim E and Redner S, 1994 Phys. Rev. E **50** 2474
[55] Derrida B, Hakim V, and Pasquier V, 1995 Phys. Rev. Lett. **75** 751
[56] Majumdar S N and Sire C, 1996 Phys. Rev. Lett. **77** 1420
[57] Lee B P and Rutenberg A D, 1997 Phys. Rev. Lett. **79** 4842
[58] Krug J, Kallabis H, Majumdar S N, Cornell S J, Bray A J and Sire C, 1997 Phys. Rev. E **56** 2702
[59] Majumdar S N and Bray A J, 1998 Phys. Rev. Lett **81** 2626
[60] Kallabis H and Krug J, 1999 Europhys. Lett. **45** 20
[61] Chakraborty D and Bhattacharjee J K, 2007 Phys. Rev. E **75** 011111
[62] Marcos-Martin M, Beysens D, Bouchand J-P, Godrèche C, and Yekutieli I, 1996 Physica D **214** 396
[63] Yurke B, Pargellis A N, Majumdar S N, and Sire C, 1997 Phys. Rev. E **56** R40
[64] Tam W Y, Zeitak R, Szeto K Y and Stavans J, 1997 Phys. Rev. Lett. **78** 1588
[65] Majumdar S, 1999 Current Science **77** 370
[66] Ray P, 2004 Phase Transitions **77**, 563
[67] Gade P M and Chin-Kun Hu, 2006 Phys. Rev. E **73** 036212
[68] Gade P M, Senthilkumar D V, Barve S and Sinha S, 2007 Phys. Rev. E **75** 066208

[69] Hinrichsen H and Koduvely H M, 1998 Eur. Phys. J. B **5** 257
[70] Albano E V and Munoz M A, 2001 Phys. Rev. E **63** 031104
[71] Fuchs J, Schelter J, Ginelli F and Hinrichsen H, 2008 J. Stat. Mech. :The. Exp. P04015
[72] Grassberger P, 2009 J. Stat. Mech. :The. Exp. P08021
[73] Menon G I, Sinha S and Ray P, 2003 Europhys. Lett. **61** 27